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Partitions and Sums and Products—Two Counterexamples

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A negative answer is provided to a question of Erdös. Specifically, a two celled partition of N is provided with the property that neither cell includes an infinite set together with all finite products and pairwise sums from that set. Also, a seven celled partition of N is provided with the property that no cell includes an infinite set together with all pairwise products and pairwise sums from that set.

1. INTRODUCTION

Erdös asked [1] whether it is always possible, given a two cell partition of N, to find one cell and an infinite subset all of whose "multilinear expressions ... (where each variable occurs only once)" are in that cell. We present here, in Section 2, a counterexample to the weaker assertion that, given a two cell partition of N, one cell has all finite products and pairwise sums from some infinite subset. We also show in Section 2 that not every finite partition of N has a cell with all pairwise sums and products from an infinite subset of the cell. (See [3] and [4] for more detailed discussion of the history of this problem.)

Section 3 consists of a brief discussion of related problems.

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2. THE COUNTEREXAMPLES

We write N for the positive integers, ω for the non-negative integers (i.e., the finite ordinals), $[A]^b$ for the set of b-element subsets of A, and fin(A) for the set of finite non-empty subsets of A. We introduce now some special notation.

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2.1. DEFINITION. (a) If $A \subseteq N$, then $FS(A) = \{\sum F: F \in \text{fin}(A)\}$, $FP(A) = \{\prod F: F \in \text{fin}(A)\}$, $PS(A) = \{x + y: \{x, y\} \in [A]^2\}$, and $PP(A) = \{x \cdot y: \{x, y\} \in [A]^2\}$.

(b) For x in N, $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are integers determined by $2^{a(x)} \leqslant x < 2^{a(x)+1}$, $2^{a(x)} + 2^{b(x)} \leqslant x < 2^{a(x)} + 2^{b(x)+1}$ (provided $x \neq 2^{a(x)}$), $2^{a(x)+1} - 2^{c(x)+1} \leqslant x < 2^{a(x)+1} - 2^{c(x)}$, and $d(x) = \max\{t: 2^t \mid x\}$.

It will be helpful to note that $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are respectively the positions of the leftmost 1, the next to leftmost 1, the leftmost 0, and the rightmost 1 of x when x is written in binary without leading zeroes. Thus if $x = 1101100$, then $a(x) = 6$, $b(x) = 5$, $c(x) = 4$, and $d(x) = 2$.

The easy proof of the following lemma is omitted.

2.2. LEMMA. *Let $x \in N$ such that $x \neq 2^{a(x)}$ and let $k \in \omega$.*

(a) *$b(x) \geqslant k$ if and only if $x \geqslant 2^{a(x)} + 2^k$.*

(b) *$b(x) \leqslant k$ if and only if $x < 2^{a(x)} + 2^{k+1}$.*

(c) *$c(x) \geqslant k$ if and only if $x < 2^{a(x)+1} - 2^k$.*

(d) *$c(x) \leqslant k$ if and only if $x \geqslant 2^{a(x)+1} - 2^{k+1}$.*

We now introduce some special sets, from which the counterexamples are constructed.

2.3. DEFINITION. (a) $A_0 = \{2^n: n < \omega\}$, $A_1 = \{x \in N \backslash A_0: a(x)$ is odd and $x < 2^{a(x)+1/2}\}$, $A_2 = \{x \in N \backslash A_0: a(x)$ is odd and $x > 2^{a(x)+1/2}\}$, $A_3 = \{x \in N \backslash A_0: a(x)$ is even and $x < 2^{a(x)+1/2}\}$, and $A_4 = \{x \in N \backslash A_0: a(x)$ is even and $x > 2^{a(x)+1/2}\}$.

(b) $B_0 = \{x \in N: a(x) - c(x) \leqslant d(x)\}$ and $B_1 = \{x \in N: a(x) - c(x) > d(x)\}$.

(c) $C_0 = \{x \in N \backslash A_0: a(x) - b(x) \leqslant d(x)\}$ and $C_1 = \{x \in N \backslash A_0: a(x) - b(x) > d(x)\}$.

(d) $D_0 = \{x \in N: x < 2^{a(x)+1}(1 - 2^{c(x)-a(x)})^{1/2}\}$ and $D_1 = \{x \in N: x \geqslant 2^{a(x)+1}(1 - 2^{c(x)-a(x)})^{1/2}\}$.

(e) $E_0 = \{x \in N \backslash A_0: x < 2^{a(x)}(1 + 2^{b(x)-a(x)+2})^{1/2}\}$ and $E_1 = \{x \in N \backslash A_0: x \geqslant 2^{a(x)}(1 + 2^{b(x)-a(x)+2})^{1/2}\}$.

(f) $F_0 = \{x \in N: a(x) - c(x)$ is even$\}$ and $F_1 = \{x \in N: a(x) - c(x)$ is odd$\}$.

(g) $G_0 = \{x \in N \backslash A_0: a(x) - b(x)$ is even$\}$ and $G_1 = \{x \in N \backslash A_0: a(x) - b(x)$ is odd$\}$.

(h) $H_0 = \{x \in N: d(x)$ is even$\}$ and $H_1 = \{x \in N: d(x)$ is odd

(i) $I_0 = \{x \in N: x$ is even$\}$ and $I_1 = \{x \in N: x$ is odd$\}$.

(j) $J_0 = A_1 \cup (A_2 \cap B_1 \cap I_0) \cup (A_3 \cap C_1 \cap I_0) \cup A_4$ and $J_1 = A_0 \cup (A_2 \cap B_0) \cup (A_2 \cap B_1 \cap I_1) \cup (A_3 \cap C_1 \cap I_1) \cup (A_3 \cap C_0)$.

(k) $K_0 = A_0 \cup A_4$, $K_1 = A_1$, $K_2 = (A_2 \cap B_0 \cap D_0 \cap F_0) \cup (A_3 \cap C_0 \cap E_0 \cap G_0)$, $K_3 = (A_2 \cap B_0 \cap D_0 \cap F_1) \cup (A_3 \cap C_0 \cap E_0 \cap G_1)$, $K_4 = (A_2 \cap B_0 \cap D_1) \cup (A_3 \cap C_0 \cap E_1)$, $K_5 = (A_2 \cap B_1 \cap H_0) \cup (A_3 \cap C_1 \cap H_0)$, and $K_6 = (A_2 \cap B_1 \cap H_1) \cup (A_3 \cap C_1 \cap H_1)$.

Note that $\{J_0, J_1\}$ and $\{K_t\}_{t<7}$ are partitions of N. We proceed now to prove several lemmas leading up to the main results. Note that if $\{x, y\} \subseteq A_1 \cup A_3$, then $a(xy) = a(x) + a(y)$ and that if $\{x, y\} \subseteq A_2 \cup A_4$, then $a(xy) = a(x) + a(y) + 1$. Consequently, we have the following easy lemma.

2.4. Lemma. (a) *If* $F \in \mathrm{fin}(A_2)$ *and* $FP(F) \subseteq A_2$, *then* $a(\Pi F) = |F| - 1 + \sum_{x\in F} a(x)$.

(b) *If* $F \in \mathrm{fin}(A_3)$ *and* $FP(F) \subseteq A_3$, *then* $a(\Pi F) = \sum_{x\in F} a(x)$.

2.5. Lemma. (a) *If* $\{x, y\} \subseteq A_2$, *then* $a(xy) - c(xy) \leqslant a(x) - c(x)$.

(b) *If* $\{x, y\} \subseteq A_3$, *then* $a(xy) - b(xy) \leqslant a(x) - b(x)$.

Proof. (a) Let $\{x, y\} \subseteq A_2$, then $a(xy) = a(x) + a(y) + 1$. Now $x < 2^{a(x)+1} - 2^{c(x)}$ and $y < 2^{a(y)+1} - 2^{c(y)}$ so $xy < 2^{a(xy)+1} - 2^{a(y)+c(x)+1} - 2^{a(x)+c(y)+1} + 2^{c(x)+c(y)} < 2^{a(xy)+1} - 2^{a(y)+c(x)+1}$. Thus, by Lemma 2.2(c), $c(xy) \geqslant a(y) + c(x) + 1 = a(xy) - a(x) + c(x)$. Therefore $a(xy) - c(xy) \leqslant a(x) - c(x)$.

(b) Let $\{x, y\} \subseteq A_3$. Then $a(xy) = a(x) + a(y)$. Since $x \geqslant 2^{a(x)} + 2^{b(x)}$ and $y \geqslant 2^{a(y)} + 2^{b(y)}$, we have $xy \geqslant 2^{a(xy)} + 2^{a(y)+b(x)} + 2^{a(x)+b(y)} + 2^{b(x)+b(y)} > 2^{a(xy)} + 2^{a(y)+b(x)}$. Thus, by Lemma 2.2(a), $b(xy) \geqslant a(y) + b(x) = a(xy) - a(x) + b(x)$.

The following lemma is an easy consequence of Lemma 2.5.

2.6. Lemma. (a) *If* $F \in \mathrm{fin}(a_2)$, $FP(F) \subseteq A_2$, *and* $x \in F$, *then* $a(\Pi F) - c(\Pi F) \leqslant a(x) - c(x)$.

(b) *If* $F \in \mathrm{fin}(A_3)$, $FP(F) \subseteq A_3$, *and* $x \in F$, *then* $a(\Pi F) - b(\Pi F) \leqslant a(x) - b(x)$.

2.7. Lemma. (a) *If* $A \in [A_2]^\omega$ *and* $FP(A) \subseteq A_2$, *then* $\{a(x) - c(x) \colon x \in A\}$ *is unbounded.*

(b) *If* $A \in [A_3]^\omega$ *and* $FP(A) \subseteq A_3$, *then* $\{a(x) - b(x) \colon x \in A\}$ *is unbounded.*

Proof. (a) Suppose instead that there are some k in N and some B in $[A]^\omega$ such that $a(x) - c(x) = k$ whenever $x \in B$. Pick d in N such that $((2^{k+1} - 1)/2^{k+1})^d \leqslant 1/2$. Pick F in $[B]^d$ and let $a = \sum_{x\in F} a(x)$. By Lemma

2.4(a), $a(\Pi F) = a + d - 1$ and hence $2^{a+d-1} \leqslant \Pi F$. On the other hand we have that for each x in F, $x < 2^{a(x)+1} - 2^{c(x)} = 2^{a(x)+1} - 2^{a(x)-k} = 2^{a(x)-k}(2^{k+1} - 1)$. Consequently $\Pi F < 2^{a-kd}(2^{k+1} - 1)^d \leqslant 2^{a-kd} \cdot 2^{kd+d-1} = 2^{a+d-1}$, a contradiction.

(b) This proof is similar, choosing d so that $((2^k + 1)/2^k)^d \geqslant 2$ and obtaining a contradiction to the fact that $\Pi F < 2^{a+1}$.

2.8. Lemma. *If $A \in [N]^\omega$, then there exists B in $[A]^\omega$ such that whenever $\{x, y\} \subseteq B$ and $x < y$, one has $d(x + y) \leqslant d(x) + 1$.*

Proof. In the event that $\{d(x): x \in A\}$ is unbounded, choose B in $[A]^\omega$ such that whenever $\{x, y\} \subseteq B$ and $x < y$ one has $d(x) < d(y)$. Then $d(x + y) = d(x)$. Otherwise choose B in $[A]^\omega$ so that whenever $\{x, y\} \subseteq B$ one has that $d(x) = d(y)$ and that $2^{d(x)+2} \mid (x - 2^{d(x)})$ if and only if $2^{d(x)+2} \mid (y - 2^{d(x)})$. (i.e., in binary x and y are both ...0100...0 or both ...1100...0). Then, whenever $\{x, y\} \subseteq B$, one has $d(x + y) = d(x) + 1$.

2.9. Lemma. (a) *If $A \in [A_2 \cap B_0]^\omega$ and $PS(A) \subseteq (A_2 \cap B_0) \cup (A_3 \cap C_0)$, then $\{a(x) - c(x): x \in A\}$ is bounded.*

(b) *If $A \in [A_3 \cap C_0]^\omega$ and $PS(A) \subseteq (A_2 \cap B_0) \cup (A_3 \cap C_0)$, then $\{a(x) - b(x): x \in A\}$ is bounded.*

Proof. (a) Suppose instead that $\{a(x) - c(x): x \in A\}$ is unbounded and choose B in $[A]^\omega$ such that $a(y) - c(y) > a(x) - c(x)$ whenever $\{x, y\} \subseteq B$ and $x < y$. Choose, by Lemma 2.8, C in $[B]^\omega$ such that $d(x + y) \leqslant d(x) + 1$ whenever, $\{x, y\} \subseteq C$ and $x < y$.

We consider first the case that $\{c(x): x \in C\}$ is bounded and pick D in $[C]^\omega$ such that $a(x) > c(y)$ whenever $\{x, y\} \subseteq D$. Pick x in D and pick y in D such that $a(y) - a(x) \geqslant d(x) + 1$. Now $y \geqslant 2^{a(y)+1} - 2^{c(y)}$ and $x \geqslant 2^{a(x)} \geqslant 2^{c(y)+1}$ so $x + y > 2^{a(y)+1}$. Also, $x + y < 2^{a(y)+1} + 2^{a(x)+1}$. Thus $a(x + y) = a(y) + 1$ and $b(x + y) \leqslant a(x)$. Since $a(x + y) = a(y) + 1$, we have $x + y \notin A_2$ and hence $x + y \in A_3 \cap C_0$. But $a(x + y) - b(x + y) \geqslant a(y) + 1 - a(x) > d(x) + 1 \geqslant d(x + y)$ so $x + y \notin C_0$.

Thus we assume that $\{c(x): x \in C\}$ is unbounded and choose D in $[C]^\omega$ such that $c(y) > a(x)$ whenever $\{x, y\} \subseteq D$ and $x < y$. Pick x in D and pick y in D such that $a(y) - c(y) \geqslant d(x) + 2$. Now $x < 2^{a(x)+1} \leqslant 2^{c(y)}$ and $y < 2^{a(y)+1} - 2^{c(y)}$ so $x + y < 2^{a(y)+1}$. Thus $a(x + y) = a(y)$ and hence $x + y \in A_2 \cap B_0$. Now $x + y > y \geqslant 2^{a(y)+1} - 2^{c(y)+1} = 2^{a(x+y)+1} - 2^{c(y)+1}$. Thus $c(x + y) \leqslant c(y)$. Hence $a(x + y) - c(x + y) \geqslant a(y) - c(y) \geqslant d(x) + 2 > d(x + y)$.

(b) This proof is similar. In fact here if $\{x, y\} \subseteq A_3$ and $a(x) + 1 < a(y)$, then $x + y \notin A_2$ so there is only one case to consider.

2.10. LEMMA. *If $A \cup PP(A) \subseteq (A_2 \cap B_1) \cup (A_3 \cap C_1)$, then $\{d(x): x \in A\}$ is bounded.*

Proof. Suppose that $\{d(x): x \in A\}$ is unbounded (in which case A is infinite) and pick B in $[A]^\omega$ such that $d(y) > d(x)$ whenever $\{x, y\} \subseteq B$ and $x < y$.

Assume first that $B \cap A_2$ is infinite. Pick x in $B \cap A_2$ and pick y in $B \cap A_2$ such that $d(y) > a(x) - c(x)$. Then $xy \notin A_3$ so $xy \in A_2 \cap B_1$. But, by Lemma 2.5, $a(xy) - c(xy) \leqslant a(x) - c(x) < d(y) \leqslant d(xy)$ and hence $xy \notin B_1$.

Similarly, if $B \cap A_3$ is infinite we pick x and y in $B \cap A_3$ such that $xy \notin (A_2 \cap B_1) \cup (A_3 \cap C_1)$.

2.11. LEMMA. *If $B \in \{A_0, A_2 \cap B_0, A_2 \cap B_1 \cap I_1, A_3 \cap C_1 \cap I_1, A_3 \cap C_0\}$, $A \in [B]^\omega$, and $FP(A) \subseteq J_1$, then $FP(A) \subseteq B$.*

Proof. It suffices to show that if $\{x, y\} \subseteq B$ and $xy \in J_1$, then $xy \in B$. The case that $B = A_0$ is trivial. For the other cases note that (1) if $\{x, y\} \subseteq A_2$, then $xy \in A_1 \cup A_2$, (2) if $\{x, y\} \subseteq A_3$, then $xy \in A_3 \cup A_4$, (3) if $\{x, y\} \subseteq I_1$, then $xy \in I_1$, and (4) $B_0 \cup C_0 \subseteq I_0$.

2.12. LEMMA. *If $B \in \{A_1, A_2 \cap B_1 \cap I_0, A_3 \cap C_1 \cap I_0, A_4\}$, then there is no A in $[B]^\omega$ such that $FP(A) \subseteq B$.*

Proof. If $B \in \{A_1, A_4\}$ we have in fact that $xy \notin B$ whenever $\{x, y\} \subseteq B$. Suppose $A \in [A_2 \cap B_1 \cap I_0]^\omega$ and $FP(A) \subseteq A_2 \cap B_1 \cap I_0$. Pick x in A and pick F in $[A \backslash \{x\}]^{a(x)-c(x)}$. Let $G = F \cup \{x\}$. Then by Lemma 2.6, $a(\Pi G) - c(\Pi G) \leqslant a(x) - c(x)$, while $d(\Pi G) \geqslant a(x) - c(x) + 1$ since each member of G is even. Thus $\Pi G \notin B_1$.

The case that $B = A_3 \cap C_1 \cap I_0$ is handled in an identical fashoin.

2.13. LEMMA. *There is no A in $[J_0]^\omega$ such that $FP(A) \subseteq J_0$.*

Proof. Suppose $A \in [J_0]^\omega$ and $FP(A) \subseteq J_0$. Choose an increasing sequence $\langle x_n \rangle_{n=1}^\infty$ in A. Let $L_0 = \{F \in \text{fin}(N): \prod_{n \in F} x_n \in A_1\}$, $L_1 = \{F \in \text{fin}(N): \prod_{n \in F} x_n \in A_2 \cap B_1 \cap I_0\}$, $L_2 = \{F \in \text{fin}(N): \prod_{n \in F} x_n \in A_3 \cap C_1 \cap I_0\}$, and $L_3 = \{F \in \text{fin}(N): \prod_{n \in F} x_n \in A_4\}$. Then $\text{fin}(N) = \bigcup_{i=0}^{3} L_i$ so by (the proof of) Corollary 3.3 of [2], one has some $i < 4$ and a disjoint sequence $\langle M_n \rangle_{n=1}^\infty$ in fin(N) such that $\bigcup_{n \in F} M_n \in L_i$ whenever $F \in \text{fin}(N)$. Letting $D = \{\prod_{t \in M_n} x_t: n \in N\}$ one obtains a contradiction to Lemma 2.12.

We are now ready for the main results.

2.14. THEOREM. *It is not the case that there exist t in $\{0, 1\}$ and A in $[J_t]^\omega$ such that $FP(A) \cup PS(A) \subseteq J_t$.*

Proof. Suppose one has such t and A and note that, by Lemma 2.13, $t = 1$. Choose B in $\{A_0, A_2 \cap B_0, A_2 \cap B_1 \cap I_1, A_3 \cap C_1 \cap I_1, A_3 \cap C_0\}$

such that $|B \cap A| = \omega$ and let $C = B \cap A$. By Lemma 2.11, $FP(C) \subseteq B$.

Assume first that $B = A_0$. Choose $x = 2^n$ and $y = 2^m$ in C such that $x > 1$ and $m > 2n$. Then $a(x+y) = m$ and $b(x+y) = d(x+y) = n$. If m is even, then $x + y \in A_3 \cup A_4$ and hence, since $x + y \in J_1$ and $x + y$ is even, $x + y \in A_3 \cap C_0$. But $a(x+y) - b(x+y) = m - n > n = d(x+y)$ so $x + y \in C_1$. Thus m is odd so that $x + y \in A_2 \cap B_0$. But $x + y = 2^m(1 + 2^{n-m}) \leqslant 2^m(1 + 2^{-2}) < 2^{m+1/2}$ so $x + y \notin A_2$.

Now assume $B = A_2 \cap B_0$. Pick x in C and, by Lemma 2.7, pick y in C such that $a(y) - c(y) > a(x) + 1$. Since $y \in B_0$, $d(y) > a(x) + 1$ and hence (no carrying occurs in the addition of x and y in binary) one has $a(x+y) = a(y)$, $c(x+y) = c(y)$, and $d(x+y) = d(x)$. Thus $a(x+y) - c(x+y) > d(x+y)$ so that $x + y \notin B_0$. Since $a(x+y) = a(y)$, $x + y \notin A_3$, and since $x + y$ is even, $x + y \notin I_1$. Thus $x + y \notin J_1$.

The case that $B = A_3 \cap C_0$ is handled in a similar fashion.

Now assume $B = A_2 \cap B_1 \cap I_1$. Pick D in $[C]^\omega$ such that, whenever $\{x, y\} \subseteq D$, $4 \mid (x - 1)$ if and only if $4 \mid (y - 1)$ (i.e., the two rightmost bits of x and y are identical). Then whenever $\{x, y\} \subseteq D$, one has $d(x + y) = 1$. Pick x in D such that $a(x) \geqslant 2$ and, by Lemma 2.7, pick y in D such that $a(y) - c(y) > a(x) + 2$. Since $d(x+y) = 1$ and $x + y > 2$, $x + y \notin A_0$. Since $x + y$ is even, $x + y \notin I_1$. Thus $x + y \in (A_2 \cap B_0) \cup (A_3 \cap C_0)$. Assume $x + y \in A_2 \cap B_0$. Then $a(x+y) = a(y)$ so $x + y > 2^{a(x+y)+1} - 2^{c(y)+1}$ and hence $c(x+y) \leqslant c(y)$. But then $a(x+y) - c(x+y) \geqslant a(y) - c(y) > a(x) \geqslant 2 > d(x+y)$ so $x + y \notin B_0$. Thus $x + y \in A_3 \cap C_0$ and hence $a(x+y) = a(y) + 1$. Now $x + y < 2^{a(y)+1} + 2^{a(x)+1} = 2^{a(x+y)} + 2^{a(x)+1}$ so $b(x+y) \leqslant a(x)$. Hence $a(x+y) - b(x+y) \geqslant a(y) - a(x) \geqslant 2 > d(x+y)$ so $x + y \notin B_0$.

Finally assume $B = A_3 \cap C_1 \cap I_1$ and, as above, pick x and y in C such that $a(x) \geqslant 2$, $a(y) - b(y) > a(x) + 2$, and $d(x+y) = 1$. Since $x + y > 2$ and $d(x+y) = 1$, $x + y \notin A_0$. Since $x + y$ is even, $x + y \notin I_1$. Now $a(y) - b(y) > a(x) + 2 > 3$ so $b(y) + 1 \leqslant a(y) - 2$. Thus $2^{a(y)} < x + y < 2^{a(y)} + 2^{b(y)+1} + 2^{a(x)+1} \leqslant 2^{a(y)} + 2^{a(y)-2} + 2^{a(y)-2} = 2^{a(y)} + 2^{a(y)-1}$. Hence $a(x+y) = a(y)$ (so that $x + y \notin A_2$) and $b(x+y) \leqslant a(y) - 2$. But then $a(x+y) - b(x+y) \geqslant 2 > d(x+y)$ so $x + y \notin C_0$. The proof is complete.

2.15. THEOREM. *There do not exist $i < 7$ and A in $[K_i]^\omega$ such that $PS(A) \cup PP(A) \subseteq K_i$.*

Proof. Suppose we have such i and A. Now $PP(A_1) \subseteq A_3 \cup A_4$ and $PP(A_4) \subseteq A_1 \cup A_2$ so $i \neq 1$ and, if $i = 0$, then $|A \cap A_0| = \omega$. In the latter case pick $x = 2^n$ and $y = 2^m$ in $A \cap A_0$ so that $m \geqslant n + 3$. Then $x + y \in A_1 \cup A_3$. Thus $i \neq 0$.

Suppose now that $i \in \{5, 6\}$. By Lemma 2.10, $\{d(x) : x \in A\}$ is bounded.

Pick distinct x and y in A such that $d(x) = d(y)$ and $d(x+y) = d(x) + 1$. Then $x + y \in H_0$ if and only if $x \in H_1$.

Assume that $i \in \{2, 3\}$ and consider first the event that $|A \cap (A_2 \cap B_0)| = \omega$. Let $B = A \cap (A_2 \cap B_0)$ and note that, by Lemma 2.9, $\{a(x) - c(x): x \in B\}$ is bounded. Pick C in $[B]^\omega$ and m such that $a(x) - c(x) = m$ whenever $x \in C$. Pick distinct x and y in C and note that $\{x, y\} \subseteq D_0$. Then $2^{a(x)+1} - 2^{c(x)+1} = 2^{a(x)+1} - 2^{a(x)-m+1} \leqslant x < 2^{a(x)+1}(1 - 2^{c(x)-a(x)})^{1/2} = 2^{a(x)+1}(1 - 2^{-m})^{1/2}$ and $2^{a(y)+1} - 2^{a(y)-m+1} \leqslant y < 2^{a(y)+1}(1 - 2^{-m})^{1/2}$. Recalling that $a(xy) = a(x) + a(y) + 1$, we have $2^{a(xy)+1} - 2^{a(xy)-m+2} + 2^{a(xy)-2m+1} \leqslant xy < 2^{a(xy)+1} - 2^{a(xy)-m+1}$. Thus $c(xy) = a(xy) - m + 1$ so we have $xy \in F_0$ if and only if $x \in F_1$. Since $xy \notin A_3$, this is a contradiction.

Thus, with $i \in \{2, 3\}$, we must have $|A \cap (A_3 \cap C_0)| = \omega$. Let $B = A \cap (A_3 \cap C_0)$ and note that, by Lemma 2.9, $\{a(x) - b(x): x \in B\}$ is bounded. Pick C in $[B]^\omega$ and m such that $a(x) - b(x) = m$ whenever $x \in C$. Pick distinct x and y in C and note that $\{x, y\} \subseteq E_0$. Then $2^{a(x)} + 2^{a(x)-m} \leqslant x < 2^{a(x)}(1 + 2^{2-m})^{1/2}$ and $2^{a(y)} + 2^{a(y)-m} \leqslant y < 2^{a(y)}(1 + 2^{2-m})^{1/2}$. Thus $2^{a(xy)} + 2^{a(xy)-m+1} + 2^{a(xy)-2m} \leqslant xy < 2^{a(xy)} + 2^{a(xy)-m+2}$. Consequently $b(xy) = a(xy) - m + 1$ so that $xy \in G_0$ if and only if $x \in G_1$. Since $xy \notin A_2$, this is a contradiction.

Finally assume that $i = 4$. Consider first the case that $|A \cap (A_2 \cap B_0)| = \omega$ and choose, as above, C in $[A \cap (A_2 \cap B_0)]^\omega$ and m such that $a(x) - c(x) = m$ whenever $x \in C$. Pick distinct x and y in C and note that $xy \notin A_3$. Note also that $\{x, y\} \subseteq D_1$. Then $2^{a(x)+1}(1 - 2^{-m})^{1/2} \leqslant x < 2^{a(x)+1} - 2^{a(x)-m}$ and $2^{a(y)+1}(1 - 2^{-m})^{1/2} \leqslant y < 2^{a(y)+1} - 2^{a(y)-m}$ so that $2^{a(xy)+1} - 2^{a(xy)-m+1} \leqslant xy < 2^{a(xy)+1} - 2^{a(xy)-m+1} + 2^{a(xy)-2m-1} < 2^{a(xy)+1} - 2^{a(xy)-m}$. Thus $c(xy) = a(xy) - m$. But $xy < 2^{a(xy)+1} - 2^{a(xy)-m+1} + 2^{a(xy)-2m} = 2^{a(xy)+1}(1 - 2^{-m} + 2^{-2m-1}) < 2^{a(xy)+1}(1 - 2^{-m})^{1/2}$ so that $xy \in D_0$.

Thus we assume $|A \cap (A_3 \cap C_0)| = \omega$ and choose C in $[A \cap (A_3 \cap C_0)]^\omega$ and m such that $a(x) - b(x) = m$ whenever $x \in C$. Pick distinct x and y in C and note that $xy \notin A_2$ and that $\{x, y\} \subseteq E_1$. Thus $2^{a(x)}(1 + 2^{2-m})^{1/2} \leqslant x < 2^{a(x)} + 2^{a(x)-m+1}$ and $2^{a(y)}(1 + 2^{2-m})^{1/2} \leqslant y < 2^{a(y)} + 2^{a(y)-m+1}$ so that $2^{a(xy)} + 2^{a(xy)-m+2} \leqslant xy < 2^{a(xy)} + 2^{a(xy)-m+2} + 2^{a(xy)-2m+2}$. Consequently $b(xy) = a(xy) - m + 2$. Now $xy \in E_1$ so $xy \geqslant 2^{a(xy)}(1 + 2^{b(xy)-a(xy)+2})^{1/2} = 2^{a(xy)}(1 + 2^{-m+4})^{1/2}$. But also $xy \in A_3$ so $xy < 2^{a(xy)} \cdot 2^{1/2}$. Thus $1 + 2^{-m+4} < 2$ so we have $m \geqslant 5$. Since $m \geqslant 5$ (in fact since $m \geqslant 3$) we have $1 + 2^{-m+2} + 2^{-2m+2} < (1 + 2^{-m+4})^{1/2}$. Thus $xy < 2^{a(xy)}(1 + 2^{-m+2} + 2^{-2m+2}) < 2^{a(xy)}(1 + 2^{-m+4})^{1/2}$, a contradiction.

3. Some Related Problems

The result of Theorem 2.15 is, in one sense, very sharp. That is it is known [4, Corollary 3.2] that given any finite partition α of N, some E in α has infinite A and B such that (1) $FS(A) \cup FP(B) \subseteq E$, (2) if $x \in FS(A)$ then x is in some infinite subset C of E with $FP(C) \subseteq E$, and (3) if $y \in FP(B)$, then y is in some infinite subset D of E with $FS(D) \subseteq E$. On the other hand, while 7 is undeniably finite, it is also much larger than 2.

3.1. *Question.* (a) Does there exist a 2 cell partition of N so that neither cell includes an infinite set A together with $PS(A) \cup PP(A)$?

(b) Does there exist a 2 cell partition of N so that neither cell includes an infinite set A together with $FS(A) \cup PP(A)$?

Finally we remark that essentially all of the finite versions remain open. (See [3] and [4] for those skimpy results which are known.) We state one of the stronger open questions.

3.3. *Question.* Given finite k and r is it true that each r cell partition of N has some cell E and some A in $[E]^k$ such that $FS(A) \cup FP(A) \subseteq E$?

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